ameter of apertures in channel wall; $\varepsilon_f$, channel porosity; $\xi$, coefficient of friction resistance; Re, Reynolds number; $\beta$, coefficient of momentum flow; $\tau$, shear stress.

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# VELOCITY AND FRICTION STRESS DISTRIBUTIONS IN THE TURBULENT BOUNDARY LAYER ON A POROUS PLATE


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The turbulent viscosity above a permeable plate is determined by using the magnitude of the maximum turbulent friction in the boundary layer. Simple dependences are obtained for the velocity and friction stress distribution.


Theoretical investigations of the turbulent boundary layer on a porous wall, whose results are suitable for engineering utilization, are based, as a rule, on a two-layer flow scheme. The Newton formula is used for the friction stress in the laminar sublayer where viscous flow predominates, while the Prandtl (or Kármán) formula is applied in the turbulent core by using the characteristic mixing path length augmented by the distance from the wall. The friction stress distribution in the boundary layer is simultaneously taken as $\tau = \tau_W + \rho_W v_W u$ [1-4], according to which $\tau$ increases monotonically over the boundary-layer thickness during blowing. The mentioned hypotheses, which are used in theoretical methods, were subjected to criticism in experimental studies [5-7]. It is shown that the magnitude of the friction stress during blowing from the wall has a maximum within the boundary layer (for $\bar{u} \approx 0.65$) but the mixing path length increases with distance from the wall only near it.

A dependence for the turbulent viscosity over a permeable plate is proposed in this paper, which agrees with recent experimental results and thus permits later investigation of the boundary layer in the presence of physicochemical processes. Let us first derive an integral relation for the momentum in a form convenient for subsequent solution of the problem. Integrating the motion and continuity equations for the boundary layer on a plate between the limits 0 and $y$ along the normal coordinate, we obtain

$$\frac{\partial}{\partial x}\left(\int_0^y \rho u^2 dy\right) = \tau - \tau_w - \rho v u, \qquad \frac{\partial}{\partial x}\left(\int_0^y \rho u dy\right) = \rho_w v_w - \rho v. \tag{1}$$

From the condition that the friction stress is zero on the outer boundary-layer limit, there follows from (1)

$$u_\infty^2 \frac{d}{dx}\int_0^\delta \rho \bar{u}(1-\bar{u})\, dy = \tau_w (1+B), \tag{2}$$


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$$\frac{\tau}{\tau_w} = 1 + B\bar{u} - (1+B)\, \frac{\bar{u}\, \frac{\partial}{\partial x}\int\limits_0^y \rho\bar{u}dy - \frac{\partial}{\partial x}\int\limits_0^y \rho\bar{u}^2 dy}{\frac{\partial}{\partial x}\int\limits_0^\delta \rho\bar{u}dy - \frac{\partial}{\partial x}\int\limits_0^\delta \rho\bar{u}^2 dy}, \qquad (3)$$

where $B = \rho_W v_W u_\infty/\tau_W$ is the blowing or suction parameter. It is shown in tests [8-10] that the velocity profiles in the turbulent part of a boundary layer with a moderate blowing intensity can be approximated by a power-law dependence of the form

$$\bar{u} = \left[\left(\int\limits_0^y \frac{\rho}{\rho_\infty}\, dy\right) \Big/ \left(\int\limits_0^\delta \frac{\rho}{\rho_\infty}\, dy\right)\right]^{1/n} = \bar{Y}^{1/n} \quad \text{or} \quad \bar{u} = \bar{y}^{1/n} \,(\text{for } \rho = \text{const}) \qquad (4)$$

with exponent n which decreases with the increase in blowing. Henceforth, let us consider the problem with just an identical relative velocity profile along the plate length [n(x) = const], which is realized for B(x) = const. Evaluating the integrals in (3) by using (4) over the whole boundary-layer thickness for this case, we obtain a simple dependence for the friction stress distribution:

$$\tau/\tau_w = 1 + B\bar{u} - (1+B)\,\bar{Y}\bar{u}^2, \qquad (5)$$

in which the assumption about the power-law nature of the velocity distribution does not enter explicitly [the deviation of the velocity distribution from a power law in a viscous sublayer influences the quantity $\tau/\tau_W$ slightly since the last member in (5) becomes commensurate with the rest only for $\bar{u} > 0.6$]. It is seen from (5) that the friction stress has a maximum within the boundary layer for gas injection ($B > 0$). In the absence of injection ($B = 0$) or during boundary-layer suction ($B < 0$), this maximum is on the wall.

Let us take the connection between the velocity and friction stress as

$$(\mu + \rho\varepsilon)\, du/dy = \tau. \qquad (6)$$

To solve (5) and (6) it is necessary to give the turbulent viscosity $\varepsilon$ as a function of y and u. We determine the form of this function from experimental results for an incompressible flow along an impermeable plate. In this case, the velocity profile in the boundary layer (including the viscous sublayer) is ordinarily represented in the dimensionless form

$$u/\sqrt{\tau_w/\rho} = f(\eta), \quad \eta = \sqrt{\tau_w/\rho}\, y/\nu. \qquad (7)$$

We obtain the following expression for $\varepsilon$ over an impermeable plate in a joint analysis of (5)-(7):

$$1 + \varepsilon/\nu = (1 - \bar{y}\bar{u}^2)/(df/d\eta). \qquad (8)$$

The main assumption of the subsequent analysis is that the form of the dependence (8) and the function $f(\eta)$ remains unchanged even in the cases of boundary-layer injection or suction; we just understand the quantity $\eta$ to be

$$\eta_m = \sqrt{\tau_m}\int\limits_0^y \left(\sqrt{\rho}/\mu\right) dy \quad \text{or} \quad \eta_m = \sqrt{\tau_m/\rho}\, y/\nu \,(\text{for } \rho,\ \mu = \text{const}), \qquad (9)$$

where $\tau_m$ is the maximum value of the turbulent friction stress in the boundary layer. The compressibility of the gas was successfully taken into account in [11] by using the dependences (8) and (9) for supersonic flow along an impermeable plate. We present certain arguments below in favor of the assumption made for an incompressible fluid flow around a permeable wall.

The experimental results presented in [6] for $\varepsilon/\nu$ near the wall as a dependence on $\eta$ with different injection or suction parameters are presented in Fig. 1 as a dependence on $\eta_m$ (to convert the coordinate $\eta$ into $\eta_m$, the value of $\tau_m/\tau_W$ was taken for injection directly from the test results [6], equal to one for $B = 0$, and calculated for suction by means of the formula (21) derived earlier for the conditions of the experiment). The introduction of the coordinate $\eta_m$ permitted generalization of all the experimental results by a single dependence in practice. The solid line corresponds to the results of a computation using (8) for $f(\eta)$ given as the dependence proposed in [11]:

$$f(\eta) = \eta\,[1 + (a\eta)^k]^{(1-n_0)/(n_0 k)}, \qquad (10)$$

which approximates the experimental results in u and $\varepsilon$ for an impermeable plate, where $n_0=7$, $k=3$, and $a = 0.0814$ are experimental constants for the Reynolds numbers $5 \cdot 10^5$–$10^7$.

The introduction of the quantity $\tau_m$ as characteristic permitted the representation of experimental results on the turbulent viscosity in the outer part of a boundary layer with and without injection in [12] as a single dependence of $\varepsilon/\sqrt{\tau_m/\rho}\,\delta$ on $\bar{y}$ with a maximum at $\bar{y}=0.4$. According to (8) and (9), the dependence of $\varepsilon/(\sqrt{\tau_m/\rho}\,\delta)^{1-1/n_0}\nu^{1/n_0}$ on $\bar{y}$ close to that found in [12] is obtained for an incompressible flow. For a compressible flow over a permeable wall, the dependence (8) is still a hypothesis not verified by experiment. Hence, we examine only incompressible flow in this paper.

Let us write (6) after having substituted (5) and (8):

$$\frac{1-\bar{y}\bar{u}^2}{df/d\eta_m} \cdot \frac{d\bar{u}}{d\bar{y}} = \frac{\tau_m \delta}{\mu u_\infty}\left(1+B\bar{u}\,\frac{1-\bar{y}\bar{u}}{1-\bar{y}\bar{u}^2}\right)(1-\bar{y}\bar{u}^2). \tag{11}$$

After dividing out the function $1-\bar{y}\bar{u}^2$, which is in the coefficient of the derivative $d\bar{u}/d\bar{y}$ in the left side of the equation, a function remains which increases over the boundary-layer thickness. A function close to $1+B\bar{u}$, which also increases monotonically over the boundary-layer thickness with injection, simultaneously remains in the right side of the equation. This means that the assumptions, used extensively in theoretical analyses, about the friction stress distribution in the form $\tau/\tau_w = 1+B\bar{u}$ and about the monotonic increase in the mixing path length over the layer thickness are completely justified, although there are no additional terms in (11). However, when writing an analogous equation for the heat flux in the presence of radiative heat transport in the boundary layer, there should be an additional term, not dependent on the rest, in its right side, and dividing both sides of the equation by $(1-\bar{y}\bar{u}^2)$ is not legitimate in this case.

To obtain the solution of (11) in analytic form, let us assume the expression $(1-\bar{y}\bar{u})/(1-\bar{y}\bar{u}^2)$ to be approximately one. A significant deviation of this fraction from one is obtained near the outer boundary-layer limit, but the maximum of this deviation reaches only 0.5 as $B\to\infty$. Taking account of the simplification made, (11) becomes

$$\frac{1}{df/d\eta_m} \cdot \frac{d\bar{u}}{d\bar{y}} = \frac{\tau_w \delta}{\mu u_\infty}(1+B\bar{u}), \tag{12}$$

whose solution under standard boundary conditions will be

$$\ln(1+B\bar{u})/\ln(1+B) = f(\eta_m)/f(\eta_{m\delta}). \tag{13}$$

For $B < 50$ this solution does not differ in practice from the numerical solution of (11). If the domain of values of $\eta_m$ is outside the limits of the viscous sublayer ($f = C\eta_m^{1/n_0}$), then the solution (13) simplifies, i.e.,

$$\ln(1+B\bar{u})/\ln(1+B) = \bar{y}^{1/n_0}. \tag{14}$$

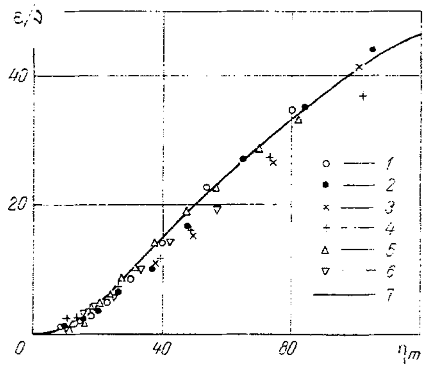


Fig. 1. Distribution of turbulent viscosity near a wall according to the results in [6]: 1) $\rho_w v_w/\rho_\infty u_\infty = 0$; 2) 0.0019; 3) 0.0038; 4) 0.0078; 5) 0.0011; 6) 0.0024; 7) computation using (8) and (10).

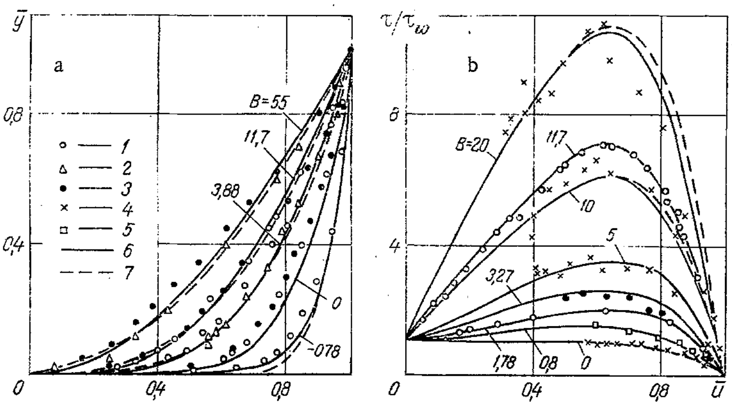

Fig. 2. Velocity (a) and friction stress (b) distributions in a turbulent boundary layer on a porous plate: 1) results from [6] for B(x) =const; 2) [6] for $\rho_W v_W$=const; 3) [13]; 4) [5]; 5) Fraser results [5]; 6) computation using (13) and (15); 7) computation using (4), (5) and (16).

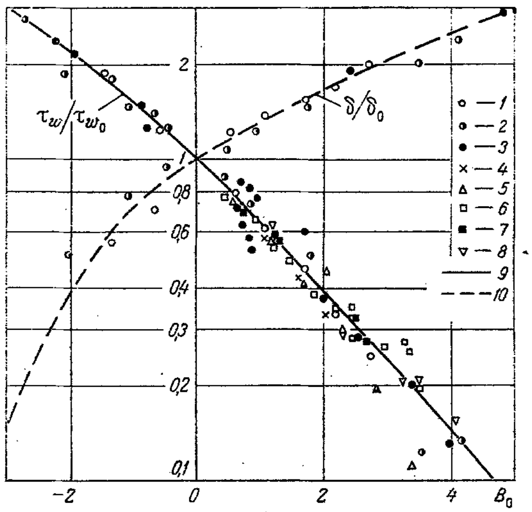

Fig. 3. Relative change in friction stress and in boundary-layer thickness on a porous plate: 1) results in [14] for B(x) =const; 2) [14] for $\rho_W v_W(x)$ =const; 3) [13]; 4) [15]; 5) Kendall results [17]; 6) [17]; 7) [16]; 8) [7]; 9) computation using (17); 10) computation using (22).

Let us substitute this expression into (5) and let us extend it to the whole range of variation of $\bar{y}$ to zero, since the last member in (5) starts to play an essential part only for $\bar{u} > 0.6$ where (14) is valid. Consequently, we obtain

$$\tau/\tau_w = 1 + B\bar{u} - (1 + B)\,\bar{u}^2\,[\ln(1 + B\bar{u})/\ln(1 + B)]^{n_0}. \tag{15}$$

In order to estimate the tolerance of the power-law approximation of the velocity profile according to (4), let us first find the exponent n by starting from the condition that the gas discharges in the boundary layer are equal:

$$G_1 = \int_0^1 \bar{u}d\bar{y} = \int_0^1 \bar{y}^{1/n}d\bar{y} = n/(n+1),$$

$$G_2 = \int_0^1 \bar{u}d\bar{y} = (1/B)\int_0^1 [(1+B)^{\bar{y}^{1/n_0}} - 1]\,d\bar{y} = (n_0/B)\int_0^1 \bar{y}^{1-1/n_0}(1+B)^{\bar{y}^{1/n_0}}\,d(\bar{y}^{1/n_0}) - 1/B.$$

When integrating the last integral by parts, we assume the function $\bar{y}^{1/n_0}$ to be approximately constant in one of the two integrands and equal to its mean value with the weight 1, i.e., $n_0/(n_0+1)$. As a result of such an integration under the condition $G_1 = G_2$ we obtain the simple formula

$$n = \frac{n_0+1}{\ln(1+B)}\left(\frac{B}{1+B}\right) - \frac{1}{1+B}. \tag{16}$$

The difference in the values of n determined as a result of exact integration of $G_2$ and according to (16) is not more than 20% for $B < 1000$. It is seen from (16) that the value of n equals $n_0$ for $B = 0$, diminishes with the increase in blowing, and grows with the increase in the suction velocity (until the viscous sublayer occupies a small part in the total turbulent layer).

Results of computing the velocity profiles by means of (13) (solid lines) and by the approximate formulas (4) and (16) (dashed lines) for different parameters B (the value of $n_0$ is taken equal to 7 here and henceforth in the computations) are presented in Fig. 2a. Results are here presented of tests 1 ($B = -0.78$; 0; 3.88; 11.7 [6]) for $B(x) = \text{const}$; 2 ($B = 4$ and 55 [6]); and 3 ($B = 0$, 11 and 55 [13]) for a constant injection velocity [because of the absence of tests for high values of $B(x) = \text{const}$; in this case the parameter B(x), which increases slightly along the length of the plate, was calculated by means of local parameters in the section under consideration]. Only those test results are here presented where the influence of the initial impermeable section is negligible. As is seen from Fig. 2a, the results of a computation using (13) and (4) agree satisfactorily with each other, as well as with the test results. A significant discrepancy between the results of computing the boundary layer near the wall with suction by means of the approximate formula (4) and the more exact formula (13) can be explained by the significant thickness of the viscous sublayer, which was realized in tests [6] with low Re numbers and is taken into account in (13).

Presented in Fig. 2b are results of computing the friction stress in the boundary layer by means of (15) and the analogous Simpson test results for $B = 1.78$ and 11.7 [6], Muzzy for $B = 0.5$, 10, and 20 [5], and Fraser and Mickley for $B = 0.8$ and 3.27 [5]. The good agreement between the experimental and computed results should be noted. Results of computing the friction stress by means of (5) by using the power-law (4) for $\bar{u}$ are presented here for $B = 10$ and 20 (dashed lines). As is seen, the law (4) permits a good approximation of (15). An expression for the relative change in friction stress on a permeable wall can be obtained from the solution (12) in the form

$$\tau_w/\tau_{w0} = [\ln(1+B)/B]^{2n_0/(n_0+1)} (\tau_m/\tau_w)^{(n_0-1)/(n_0+1)} (\delta_0/\delta)^{2/(n_0+1)}. \tag{17}$$

The determination of the quantity $(\tau_m/\tau_w)$ in (17) explicitly from the dependence (15) is difficult because it is transcendental. This difficulty can be overcome by using the dependence (5) and the power-law function (4) for $\bar{u}$ with an exponent n determined by (16) and by knowing that the value of $\tau_m$ is realized for $\bar{u} > 0.6$. In this case, we write the expression for the turbulent friction stress as

$$\tau_t/\tau_w = 1 + B\bar{u} - (1+B)\bar{u}^{n+2} - \mu u_\infty/(\delta n \tau_w \bar{u}^{n-1}). \tag{18}$$

The last member in (18), corresponding to molecular friction, can be neglected during injection ($B > 1$, $\bar{u} > 0.6$) as compared to the rest. We then obtain from the condition $d\tau_t/d\bar{u} = 0$

$$\bar{u}_m = \left(\frac{B}{1+B}\cdot\frac{1}{n+2}\right)^{1/(n+2)}, \quad \frac{\tau_m}{\tau_w} = 1 + B\bar{u}_m\frac{n+1}{n+2}, \tag{19}$$

where $\bar{u}_m$ is the relative velocity at which the maximum friction is realized. For an impermeable wall ($B = 0$), we obtain from (18)

$$\bar{u}_m = \left[\frac{\mu u_\infty (n_0-1)}{\delta_0 \tau_{w0} n_0 (n_0+2)}\right]^{1/(2n_0+1)}, \quad \frac{\tau_m}{\tau_{w0}} = 1 - \bar{u}_m^{n_0+2} - \frac{\mu u_\infty}{\delta_0 \tau_{w0} n_0 \bar{u}_m^{n_0-1}}. \tag{20}$$

According to the computations for $Re_\delta > 10^4$, the last two members in (20) are small compared to one. Hence, for an impermeable wall the choice between the maximum turbulent friction stress and the friction stress on a wall is equivalent.

For sufficiently intensive boundary-layer suction from the wall ($B < -0.5$, $\bar{u} < 0.9$), the third member in (18) can be neglected in comparison with the rest. We consequently obtain

$$\bar{u}_m = [-(n-1)\mu/(n\rho_w v_w \delta)]^{1/n}, \quad \tau_m/\tau_w = 1 + B\bar{u}_m n/(n-1). \tag{21}$$

As computations using (19)-(21) have shown, the value of $\bar{u}_m$ is within the limits of 0.6-0.75 as B varies between 1000 and −0.8.

The relative change in boundary-layer thickness as a result of injection or suction, which enters into (17), can be determined in terms of the change in thickness of the loss of momentum (2). When using the power-law (4) for the velocity and in the absence of an initial impermeable section, we obtain from (2)

$$\frac{\delta}{\delta_0} = \left[\frac{\tau_w(1+B)}{\tau_{w0}}\right] \frac{(n+1)(n+2)/n}{(n_0+1)(n_0+2)/n_0}. \tag{22}$$

Results of computing the relative change in friction stress on a permeable wall by means of (17), (19), (21), and (22) as a function of the parameter $B_0 = B(\tau_W/\tau_{W_0})$ (solid line) and the corresponding results of different tests are represented in Fig. 3. As is seen, the computational dependence describes the experimental results presented in Fig. 3 satisfactorily. Also presented in Fig. 3 is a comparison between the results of computing the relative change in boundary-layer thickness with gas injection and suction from the wall by means of (22) and the corresponding test results of Simpson [6] and Mickley [13]. The results represented are in satisfactory agreement.

## NOTATION

x, y, u, v, coordinates and mean gas velocity components along and across the plate, respectively; $\rho$, $\mu$, $\nu$ and $\varepsilon$ density, viscosity, and molecular and turbulent kinematic viscosities of the gas; $\delta$, boundary-layer thickness; $\vartheta$, thickness of the loss of momentum; $\bar{y} = y/\delta$, $\bar{u} = u/u_\infty$, relative coordinate and velocity; $\tau$, friction stress; $\tau_m$, $u_m$, maximum value of the turbulent friction stress and velocity at which it is realized; $B = \rho_W v_W u_\infty/\tau_W$, $B_0 = \rho_W v_W u_\infty/\tau_{W0}$, blowing or injection parameters; $\eta = \sqrt{\tau_w/\rho_w}\, y/\nu$, $\eta_m = \sqrt{\tau_m/\rho}\, y/\nu$ dimensionless distances from the wall; n, exponent. Indices: $\infty$, outside the boundary layer; $\delta$, on the boundary-layer limit; w, on the wall; 0, in the absence of blowing or suction.

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